Section - A

1. Attempt all questions.  

(a) Show that the set of all even numbers is a countably infinite set.

(b) Consider the following relations on the set $A = \{1, 2, 3, 4\}$, determine whether the following relations are reflexive, symmetric, anti-symmetric or transitive.

   (i) $R = \{(1, 3), (1, 1), (3, 1), (1, 2), (3, 3), (4, 4)\}$

   (ii) $R = A \times A$

(c) What is a composite relation?

(d) Consider the set $Z_4 = \{0, 1, 2, 3\}$, together with multiplication modulo 4. Is $Z_4$ a group?

(e) Consider statement ‘$p$: If it rains the street get wet’. Consider statement ‘$q$: The streets are wet.’ Can we claim that if $q$ is true then $p$ is true?’

(f) Is it necessary for a graph having Hamiltonian cycle to have an Eulerian tour? Give proof or a counter-example.

(g) Draw a planar graph with 6 vertices with each vertex having degree more than or equal to 3.

(h) Write Pre-order and Post-order traversal of the following binary tree.

   ![Binary Tree]

(i) What are the properties of a complemented Lattice?

(j) Prove that the difference of some two integers out of $(n + 1)$ integers divides $n$. 

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Section – B

Attempt three questions: \(10 \times 3 = 30\)

2. Attempt any 3 out of 5:
   (a) Let A, B, C be subsets of universal set U. Given that
   \[A \cap B = A \cap C \text{ and } \overline{A} \cap B = \overline{A} \cap C\]
   Is it necessary that \(B = C\)? Justify your answer.
   (b) What is a Partial Order? Consider the set \(Z\) of integers. Define ‘\(a \leq b\)’ by ‘\(b = a\)’ for some positive integer \(r\). Show that \(R\) is a partial order on \(Z\).
   (c) Prove Lagrange’s theorem that states “for any finite group \(G\), the order (number of elements) of every subgroup \(H\) of \(G\) divides the order of \(G\).”
   (d) What is a Tautology and what is a Contradiction? Draw truth tables for \(\overline{p} \lor q\), \(p \rightarrow q\), \(p \leftrightarrow q\), \(\overline{p} \lor q\). Which of them are/is a tautology?
   (e) Prove that: Let \((P, \leq)\) be a partially ordered set. Suppose the length of the longest anti-chain in \(P\) is \(n\). Then the elements in \(P\) can be partitioned into \(n\) disjoint chains.

Section – C

Attempt all questions. In each question attempt either part ‘A’ or part ‘B’. \(10 \times 5 = 50\)

3. Part A: (1) A set \(S\) consists of two types of elements: Type 1 and Type 2. Both Type 1 and Type 2 subsets are non-empty. A relation \(R\) on \(S\) is defined such that \((a, b) \in R\) only if \(a\) and \(b\) are of different types. Can such a relation be reflexive, symmetric, transitive, equivalence? Explain your answer.
   (2) Prove by induction that the partial sum of the terms of Fibonacci sequence is
   \[F_0 + F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1\]
   In a Fibonacci sequence every element is sum of previous two elements, with first two elements as 0 and 1. The Fibonacci sequence is given as 0 1 1 2 3 5 8 13 21 34...
   OR

Part B: (1) Prove or disprove: \(P(A \cup B) = P(A) \cup P(B)\), where \(P(X)\) denotes the power set of set \(X\). A power set of \(X\) is a set containing all subsets of \(X\).
   (2) (i) Among the integers from 1 to 300, how many are not divisible by 2, nor by 3, nor by 5.
   (ii) Among the integers from 1 to 300, how many are divisible by 2 and 3 but not by 5.
4. Part A: Consider the set $S = \{1, \omega, \omega^2\}$, where $\omega$, $\omega^2$ are complex cube roots of unity. If $\ast$ denotes the multiplication operation, show that the algebraic structure $(S, \ast)$ forms an Abelian group.

OR

Part B: Prove that a subgroup of a cyclic group is cyclic.

5. Part A: (a) Show that if the sum of the degrees for each pair of vertices of a graph $G$ with $n$ vertices is $n$ or larger, then there exists a Hamiltonain cycle in $G$.

OR

Part B: (1) Let $G$ be a disconnected graph with $k$ connected components. How many minimum number of edges need to be added to make $G$ connected?

(2) Show that if all the internal vertices of a tree have same degree, then it has total odd number of vertices or give a counter-example.

6. Part A: (1) Let set $S = \{a, b, c, d, e\}$ and set $P$ be set of partitions of $S$ such that $P = \{P_1, P_2, P_3, P_4\}$. 

\[ P_1 = \{(a, b, c), (d, e)\}, \quad P_2 = \{(a, b), (c, d, e)\}, \quad P_3 = \{(a, b, c, d, e)\} \]

and

\[ P_4 = \{(a), (b), (c), (d), (e)\}. \]

A partial order on $P$ is defined such that $P_i \leq P_j$ if and only if all elements of $P_i$ are subset of elements of $P_j$.

(i) Express the partial order using a Hasse diagram.

(ii) Is this partial order a lattice? Explain your answer.

(2) Prove that in a Boolean Lattice $(b \leq a)$ if and only if $(\overline{a} \leq \overline{b})$.

OR

Part B: (1) Let $(A, \leq)$ be a distributive lattice. Show that, if

\[ a \land x = a \land y \quad \text{and} \quad a \lor x = a \lor y \]

for some $a$, then $x = y$.

(2) Let $a$, $b$, $c$ be elements in a non-distributive lattice $(A, \leq)$. Show that if $a \leq b$, then $a \lor (b \land c) \leq b \land (a \lor c)$.
7. Part A: (1) Given the premise show the truth or falsity of the following conclusion by drawing the truth table.

**Premise:**
1. If it rains, streets get wet
2. If streets get wet, Accidents happen
3. It has not rained

**Conclusion:** Streets are not wet.

(2) Prove if $p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \land (q \rightarrow p)$?

**OR**

Part B: (1) Prove the validity of the following argument:

If I study, then I will not fail in Mathematics.

If I do not play Basketball, then I will study.

I failed in Mathematics.

I must have played Basketball.

(2) Prove if $\neg p \rightarrow q$ is equivalent to $\neg q \rightarrow p$?